13.4 More Integral Applications

In this section, we explore two more integral applications to business:

- Income flow
- Consumer/Supplier Surplus

Income Flow

If total income from a continuous income stream has an **annual rate** of flow given by $r(t)$, then the total income in *k* years is

$$I(k) = \int_0^k r(t)dt.$$

This formula applies if income comes in

1. "spread out" (continuous) throughout the whole year, and
2. with an annual rate $r(t)$.

Example:

1. Constant annual rate

$$r(t) = 4000 \quad \text{dollars/year}$$

What is total income in the first 5 years

$$I(5) = \int_0^5 4000\,dt$$

$$= 4000t\Big|_0^5$$

$$= 4000(5) = \boxed{\$20{,}000}$$

DIDN'T REALLY NEED CALCULUS TO FIGURE THIS OUT!

2. Linearly increasing rate

$$r(t) = 3000 + 250t \quad \text{dollars/year}$$

What is total income in the first 8 years?

\$3000 in first year (spread out)
3250 in second year (spread out)
3500 in third year
etc.
⋮

$$\int_0^8 3000 + 250t \, dt$$

$$= 3000t + 125t^2 \Big|_0^8$$

$$= (3000 \cdot (8) + 125(8)^2) - (0)$$

$$= \boxed{\$32{,}000}$$

3. Exponential rate (most common, *i.e.* bank account and investments)

$$r(t) = 800e^{0.05t} \quad \text{dollars/year}$$

Aside: In this model, \$800/year is the initial rate at which income is coming ir and it is increasing at 5% per year (spread out throughout the year).

What is total income in the first 6 years

$$\int_0^6 800e^{0.05t} \, dt$$

$$\frac{800}{0.05} e^{0.05t} \Big|_0^6$$

$$16000 \, e^{0.05t} \Big|_0^6$$

$$16000 \left(e^{0.05(6)} - e^{0.05(0)}\right)$$

$$16000 \left(e^{0.3} - 1\right)$$

$$\approx \boxed{\$5597.74}$$

Consumer/Supplier Surplus

Recall:

Demand Curve: Relates price per item, *p*, to the number of items, *x*, consumers will buy at that price.

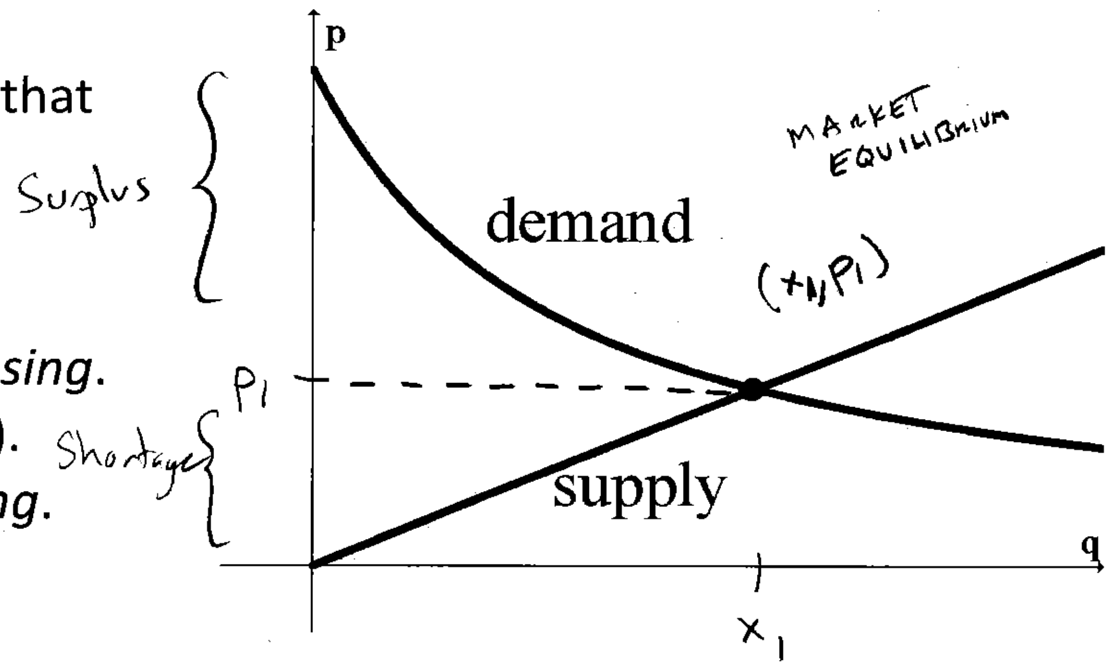
Supply Curve: Relates price per item, *p*, to the number of items, *x*, that manufacturers are willing to sell at that price.

And remember:

The demand curve is always *decreasing*. (Price goes up, demand goes down).

The supply curve is always *increasing*. (Price goes up, supply goes up).

Market equilibrium is the quantity and price at which supply and demand intersect. At this price, the amount produced and sold will be equal (no *shortage* and no *surplus*)



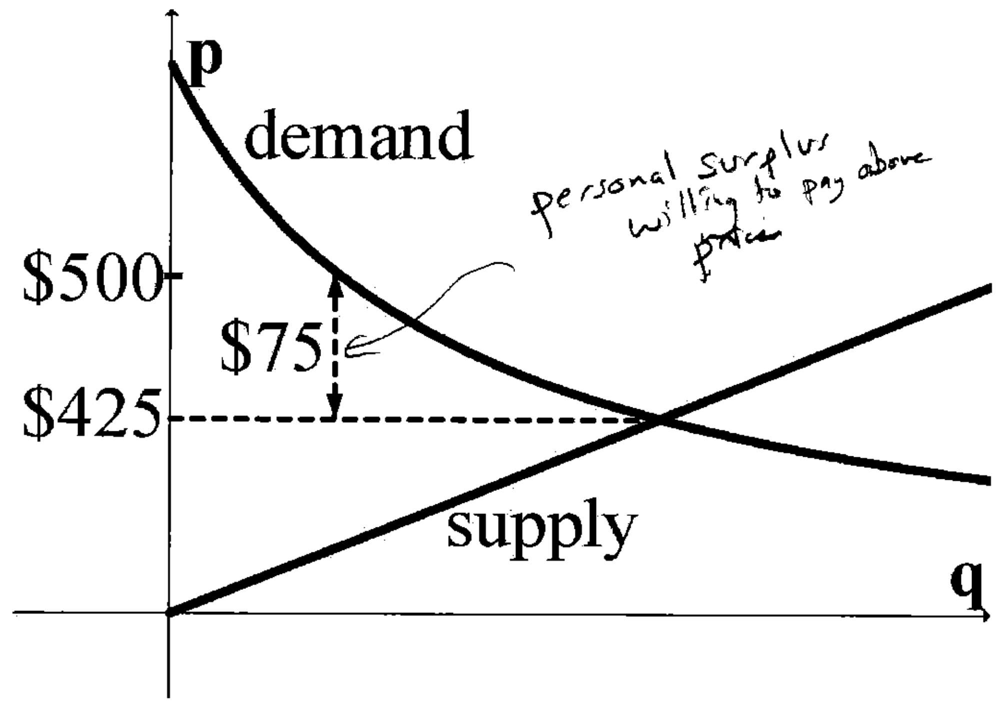
Consumer Surplus

Idea: You go to the store to buy some new piece of technology. You go ready to spend $500, but when you get to the store you find that it is on sale for $425.

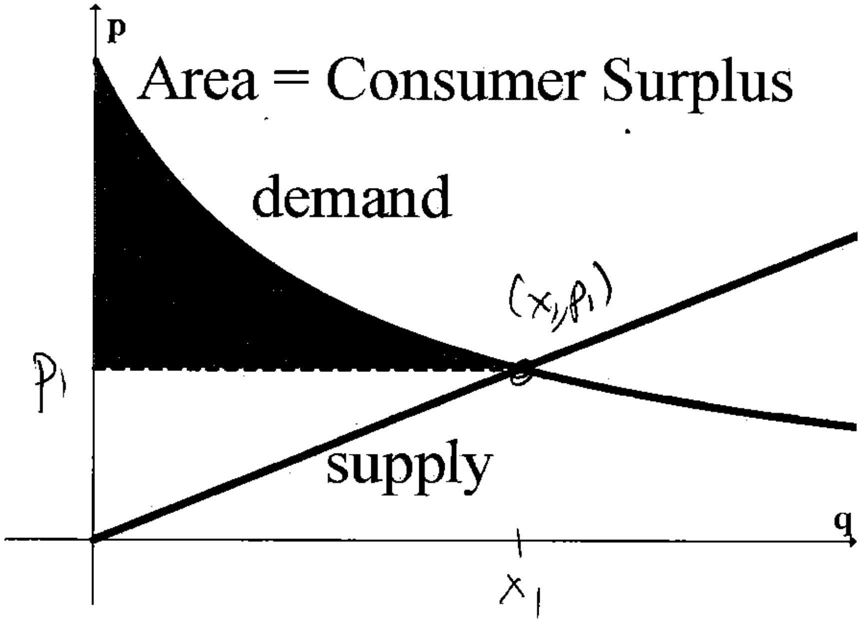
From your perspective:
You happily saved $75.
From the store's perspective: They could have made $75 more from you if they would have known how desperate you were for that new piece of technology.

We could say that you have a personal consumer surplus of $75. If $425 is the market equilibrium price, then your willingness to buy at $500 means you are part of the demand curve that comes before the equilibrium.

And you can visualize this $75 amount as the distance between the demand curve and the horizontal equilibrium line at $425.



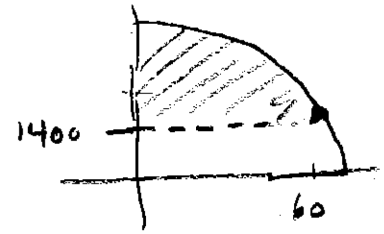
The sum of all moneys that some consumers are willing to *pay over the equilibrium* for a given product is called **Consumer Surplus**. It is given by the area of the region below:



If demand is given by $p = f(x)$ and equilibrium is at $(x, p) = (x_1, p_1)$, then

$$\text{CS} = \int_0^{x_1} f(x)dx - p_1 x_1$$

Example: If the demand curve is $p = 5000 - x^2$, and if market equilibrium is at $(x, p) = (60{,}1400)$ then find consumer surplus.



$$\underbrace{\int_0^{60} 5000 - x^2 dx} - \underbrace{1400 \cdot 60}_{TR(60) = \text{AREA OF RECTANGLE}}$$

$$\underbrace{5000x - \tfrac{1}{3}x^3 \Big|_0^{60}} - 84{,}000$$

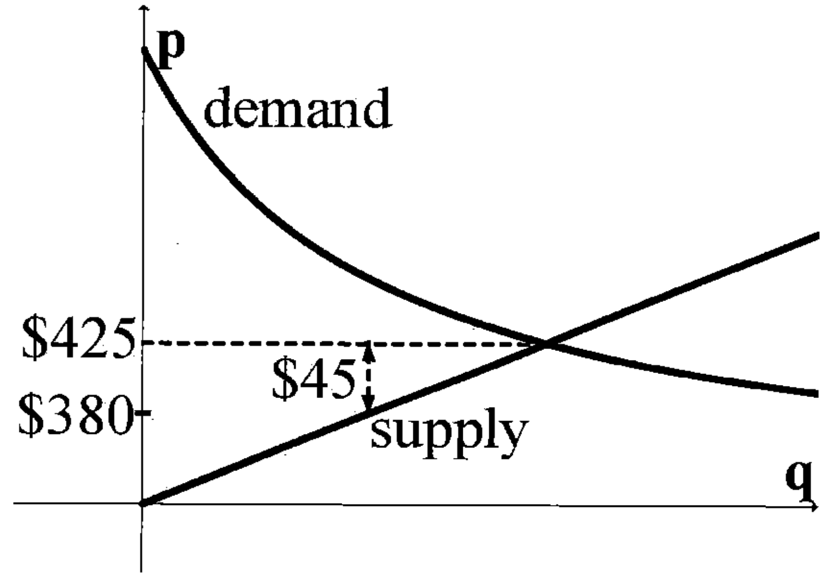
$$\left(5000(60) - \tfrac{1}{3}(60)^3\right) - (0) - 84{,}000$$

$$= \$228{,}000 - \$84{,}000$$

$$= \$144{,}000$$

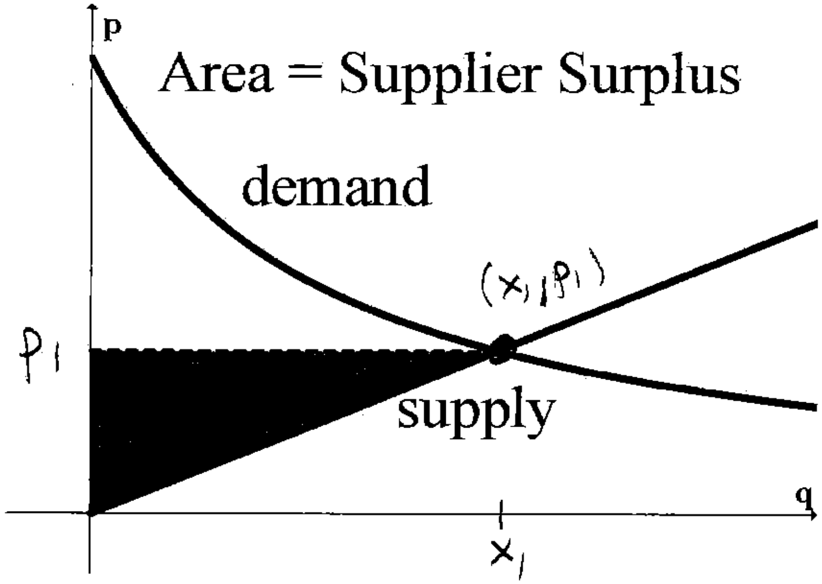
Producer (Supplier) Surplus

Idea: Assume a supplier produced and sold the same bit of technology from my earlier story. They had planned to sell it for $380 at a different store and as a result had produced fewer quantities than most manufacturers (or you could say they were willing to sell for less than market equilibrium).



But it turned out that the equilibrium price is $425 and they can sell for a $45 *surplus* from what they had originally planned (or you could say they left $45 per item "on the table").

The sum of all moneys for suppliers willing to sell for less than *equilibrium* for a given product is called **Producer Surplus**. It is given by the area of the region below:



If supply is given by $p = g(x)$ and equilibrium is at $(x, p) = (x_1, p_1)$, then

$$\text{PS} = p_1 x_1 - \int_0^{x_1} g(x)dx$$

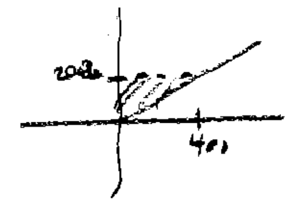
Example: If the supply curve is $p = 2 + 0.5x$, and if market equilibrium is at $(x, p) = (400{,}202)$ then find supplier surplus.

$$\underbrace{202 \cdot 400}_{TR(400)} - \int_0^{400} 2 + 0.5x\,dx$$

$$= \$80{,}800 - \underbrace{\left[2x + 0.25x^2 \Big|_0^{400} \right]}$$

$$= 80{,}800 - 40{,}800$$

$$= \boxed{\$40{,}000}$$



Example (Problems 7 and 11 from HW)

Given Demand: $p = \frac{48}{x+2}$

Supply: $p = 3 + 0.1x$

Find consumer and producer surplus under pure competition (meaning at market equilibrium).

Step 1: Find market equilibrium.

$$Step\ 2: CS = \int_0^{x_1} f(x)dx - p_1x_1$$

$$Step\ 3: PS = p_1x_1 - \int_0^{x_1} g(x)dx$$

STEP 1 $\quad 3 + 0.1x = \frac{48}{x+2}$

$(x+2)(3+0.1x) = 48$

$\Rightarrow \quad 0.1x^2 + 3x + 0.2x + 6 = 48$

$0.1x^2 + 3.2x - 42 = 0$

$x^2 + 32x - 420 = 0$

QUAD FORMULA or $(x-10)(x+42) = 0$

$\boxed{x = 10}$ / $x = -42$

$x = 10 \Rightarrow p = 3 + 0.1(10) = 4$ ✓

$p = \frac{48}{(10+2)} = 4$ ✓

$\boxed{x_1 = 10, \; p_1 = 4 \quad \text{MARKET EQUILIBRIUM}}$

$$CS = \int_0^{10} \frac{48}{x+2}dx - 4 \cdot 10$$

$$= 48\ln(x+2)\Big|_0^{10} - 40$$

$$= 48\ln(12) - 48\ln(2) - 40$$

$$\approx 86.0044545\text{2} - 40$$

$$= \boxed{\$46.00}$$

$$PS = 4 \cdot 10 - \int_0^{10} 3 + 0.1x\,dx$$

$$= 40 - \left[3x + 0.05x^2\Big|_0^{10}\right]$$

$$= 40 - [(3(10) + 0.05(10)^2) - 0]$$

$$= 40 - 35$$

$$= \boxed{\$5}$$